Sample Exam (0 points)

September 26, 2019 Name: ____________________________

No notes, calculators, or other aids are allowed. Read all directions carefully and write your answers in the space provided. To receive full credit, you must show all of your work.

1. *Math mode* is used to display mathematical content in LaTeX, and there are two main forms of math mode: *display mode* and *inline mode*. Question **??** uses *display mode*, which centers the math content on its own line. Question **??** uses *inline mode* to render the math content within a line of text.

Algebra — Algebraic Operations/Calculations

2022 - Trial Exam 2 - Section A - Question 6

If $f(x+1) = x^2 + 3x$ then $f(x-1)$ is equal to

A. $x^2 + 3x - 2$

B. $x^2 - x - 2$

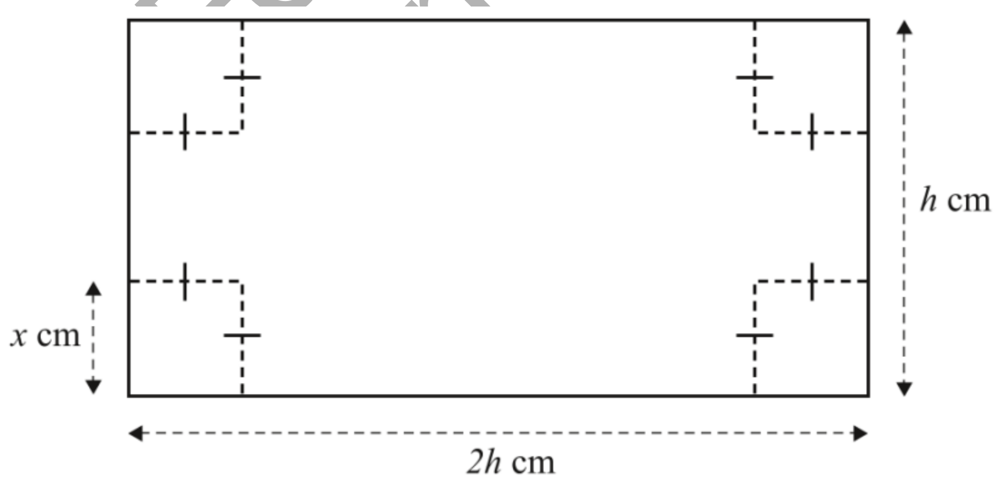
C. $x^2 + x - 2$

D. $x^2 + 3x - 1$

E. $x^2 + 5x + 4$

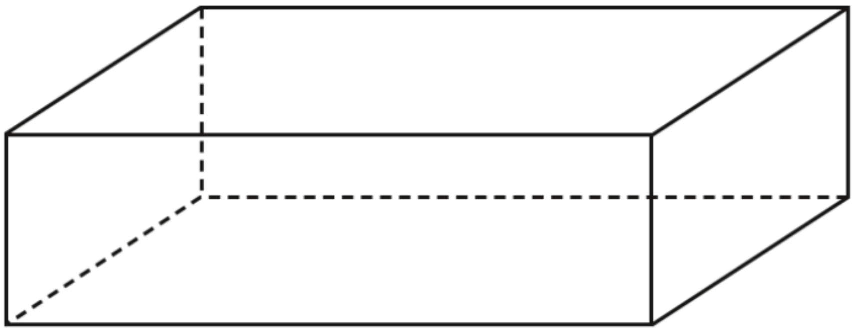
2021 - Exam 2 - Section B - Question 1 - Part e

A rectangular sheet of cardboard has a width of h centimetres. Its length is twice its width. Squares of side length x centimetres, where $x > 0$, are cut from each of the corners, as shown in the diagram below.



The sides of this sheet of cardboard are then folded up to make a rectangular box with an open top, as shown in the diagram below.

Assume that the thickness of the cardboard is negligible and that $V_{box} > 0$.



A box to be made from a sheet of cardboard with $h = 25cm$

e. Waste minimisation is a goal when making cardboard boxes.

Percentage wasted is based on the area of the sheet of cardboard that is cut out before the box is made.

Find the percentage of the sheet of cardboard that is wasted when $x = 5$.

2019 - NH Exam 2 - Section A - Question 6

Let $f : [0, \infty) \to R, f(x) = x^2 + 1$.

The equation $f(f(x)) = \frac{185}{16}$ has real solution(s)

A. $x = \pm\dfrac{\sqrt{13}}{4}$

B. $x = \dfrac{\sqrt{13}}{4}$

C. $x = \pm\dfrac{\sqrt{13}}{2}$

D. $x = \dfrac{3}{2}$

E. $x = \pm\dfrac{3}{2}$

2019 - NH Exam 1 - Section A - Question 3 - Part b

b. Show that $\dfrac{1}{2}\left(\dfrac{1}{x-\sqrt{3}} + \dfrac{1}{x+\sqrt{3}}\right) = \dfrac{x}{x^2-3}$.

2019 - Exam 2 - Section A - Question 2

The set of values of k for which $x^2 + 2x - k = 0$ has two real solutions is

A. $\{-1, 1\}$

B. $(-1, \infty)$

C. $(-\infty, -1)$

D. $\{-1\}$

E. $[-1, \infty)$

2018 - NH Exam 2 - Section A - Question 13

The function f has the property $f(2x) = (f(x))^2 - 2$ for all real numbers x. A possible rule for the function $f(x)$ is

A. $\dfrac{1}{x^2+4}$

B. $\cos(x)$

C. $2\log_e(x^2+1)$

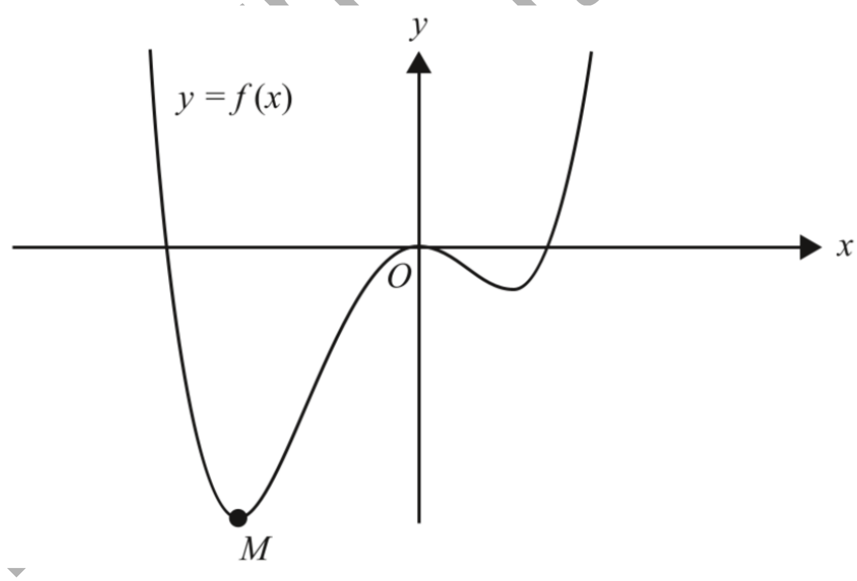
D. $e^x + e^{-x}$

E. x^2

2018 - Exam 2 - Section B - Question 1

Question 1 (13 marks)

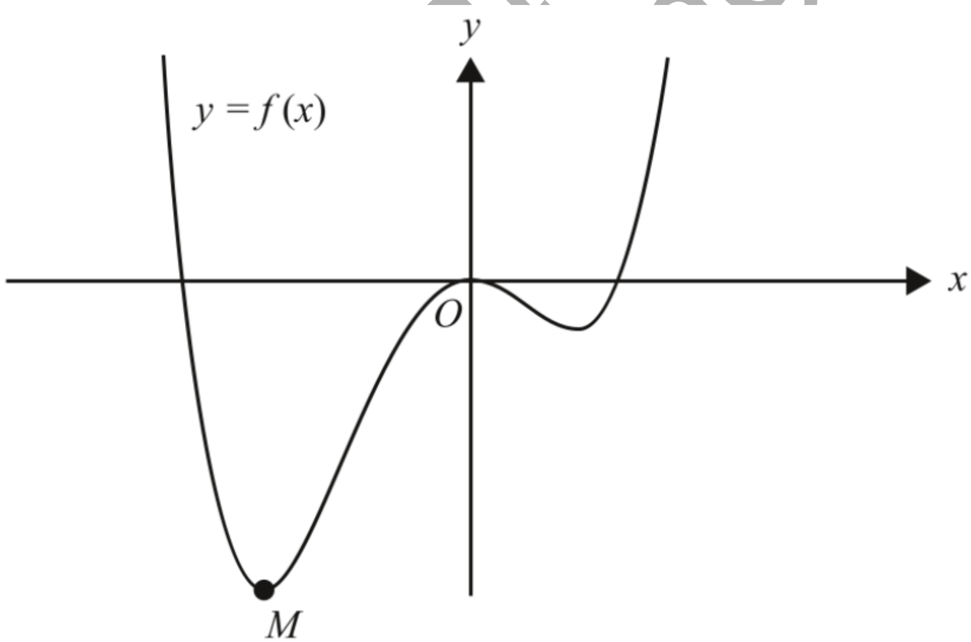
Consider the quartic $f : R \to R, f(x) = 3x^4 + 4x^3 - 12x^2$ and part of the graph of $y = f(x)$ below



a. Find the coordinates of point M, at which the minimum value of the function f occurs.

b. State the values of $b \in R$ for which the graph of $y = f(x) + b$ has no x-intercepts.

Part of tangent, l, to $y = f(x)$ at $x = -\dfrac{1}{3}$ is shown below



c. Find the equation of tangent l.

d. The tangent l intersects $y = f(x)$ at $x = -\frac{1}{3}$ and at two other points

State the x-values of the two other points of intersection. Express your answers in the form $\frac{a \pm \sqrt{b}}{c}$, where a, b and c are integers.

e. Find the total area of the regions bounded by the tangent l and $y = f(x)$. Express your answer in the form $\frac{a\sqrt{b}}{c}$, where a, b and c are positive integers.

Let $p : R \to R, p(x) = 3x^4 + 4x^3 + 6(a-2)x^2 - 12ax + a^2, a \in R$.

f. State the value of a for which $f(x) = p(x)$ for all x.

g. Find all solutions to $p'(x) = 0$, in terms of a where appropriate.

h. **i.** Find the values of a for which p has only one stationary point.

ii. Find the minimum value of p when $a = 2$.

iii. If p has only one stationary point, find the values of a for which $p(x) = 0$ has no solutions.

2017 - NH Exam 2 - Section A - Question 17

A function f satisfies the relation $f\left(x^2\right) = f(x) + f(x+2)$. A possible rule for f is

A. $f(x) = \sqrt{x+2}$

B. $f(x) = x + 2$

C. $f(x) = \log_{10}(x-1)$

D. $f(x) = \frac{1}{2}\left(x^2 - 1\right)$

E. $f(x) = \frac{1}{x-1}$

2017 - NH Exam 2 - Section B - Question 4 - Part d-ei

Let $f : [0, 16] \to R, f(x) = 4\sqrt{x} - x$.

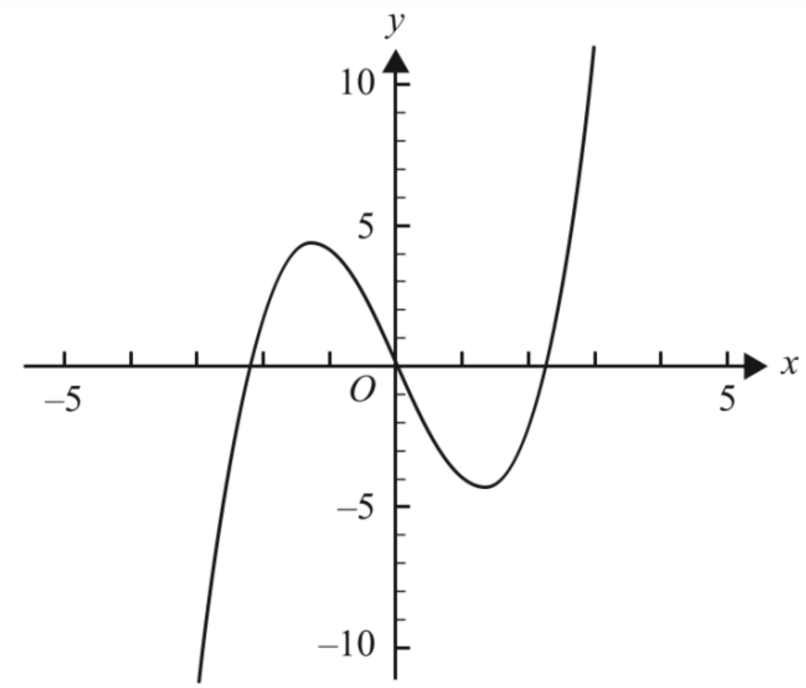
d. Show that $b = (\sqrt{a} - 4)^2$.

e. A rectangle is formed with vertices $(a, 0), (b, 0), (b, f(b))$ and $(a, f(a))$.

i. Find the area of this rectangle in terms of a.

2017 - Exam 2 - Section 2 - Question 1 - Part b (ii)

$f : R \to R, f(x) = x^3 - 5x$. Part of the graph of f is shown below



b. $A(-1, f(-1))$ and $B(1, f(1))$ are two points on the graph of f.

ii. Find the distance AB.

2017 - Exam 2 - Section 2 - Question 1 - Part c (i)

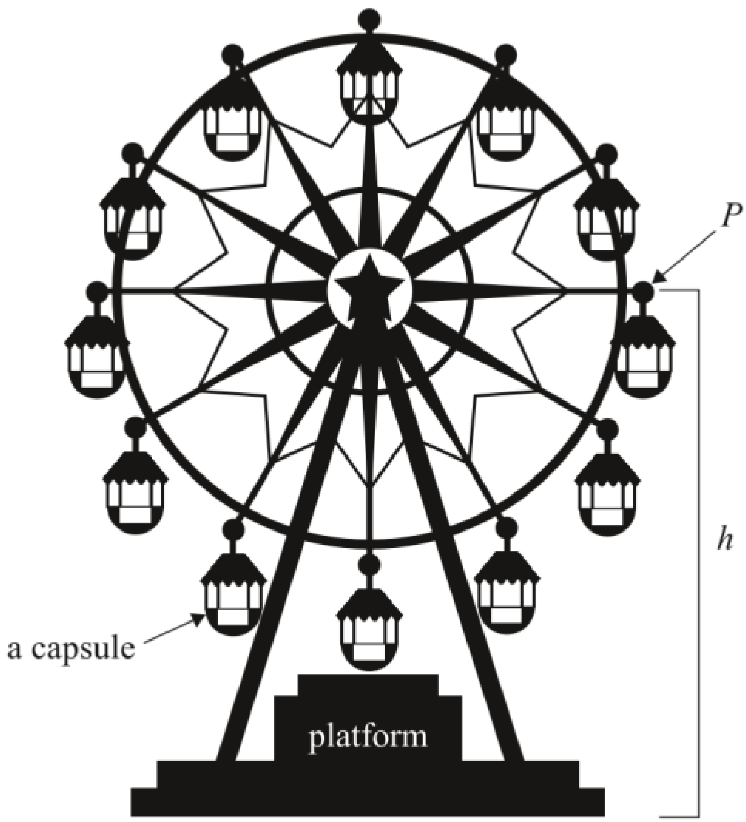
Let $g : R \to R, g(x) = x^3 - kx, k \in R^+$.

c. Let $C(-1, g(-1))$ and $D(1, g(1))$ be two points on the graph of g.

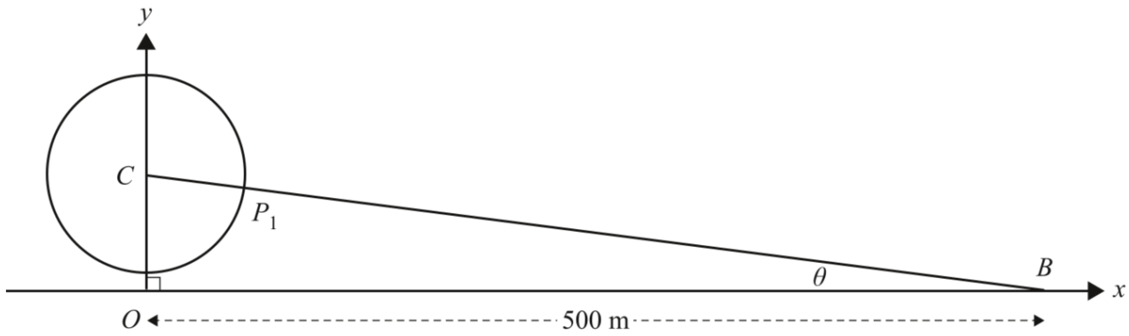
i. Find the distance CD in terms of k.

2017 - Exam 2 - Section 2 - Question 2 - Part d

Sammy visits a giant Ferris wheel. Sammy enters a capsule on the Ferris wheel from a platform above the ground. The Ferris wheel is rotating anticlockwise. The capsule is attached to the Ferris wheel at point P. The height of P above the ground, h, is modelled by $h(t) = 65 - 55\cos\left(\frac{\pi t}{15}\right)$, where t is the time in minutes after Sammy enters the capsule and h is measured in metres. Sammy exits the capsule after one complete rotation of the Ferris wheel.



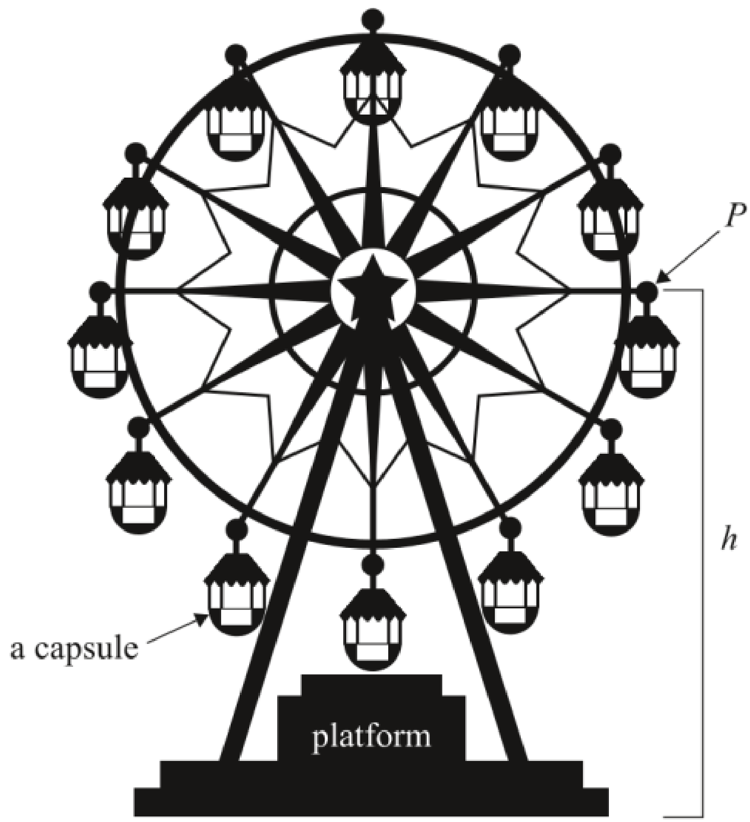
As the Ferris wheel rotates, a stationary boat at B, on a nearby river, first becomes visible at point P_1. B is 500 m horizontally from the vertical axis through the centre C of the Ferris wheel and angle $CBO = \theta$, as shown below.



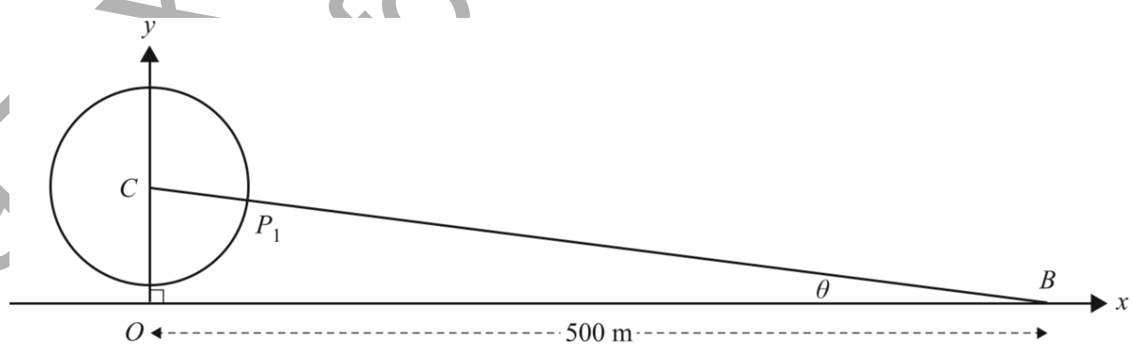
d. Find θ in degrees, correct to two decimal places.

2017 - Exam 2 - Section 2 - Question 2 - Part g-h

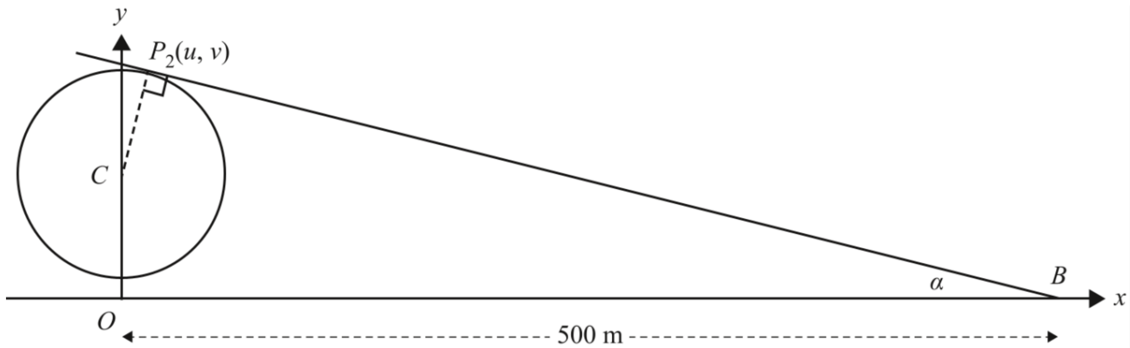
Sammy visits a giant Ferris wheel. Sammy enters a capsule on the Ferris wheel from a platform above the ground. The Ferris wheel is rotating anticlockwise. The capsule is attached to the Ferris wheel at point P. The height of P above the ground, h, is modelled by $h(t) = 65 - 55\cos\left(\frac{\pi t}{15}\right)$, where t is the time in minutes after Sammy enters the capsule and h is measured in metres. Sammy exits the capsule after one complete rotation of the Ferris wheel.



As the Ferris wheel rotates, a stationary boat at B, on a nearby river, first becomes visible at point P_1. B is 500 m horizontally from the vertical axis through the centre C of the Ferris wheel and angle $CBO = \theta$, as shown below.



As the Ferris wheel continues to rotate, the boat at B is no longer visible from the point $P_2(u, v)$ onwards. The line through B and P_2 is tangent to the path of P, where angle $OBP_2 = \alpha$.

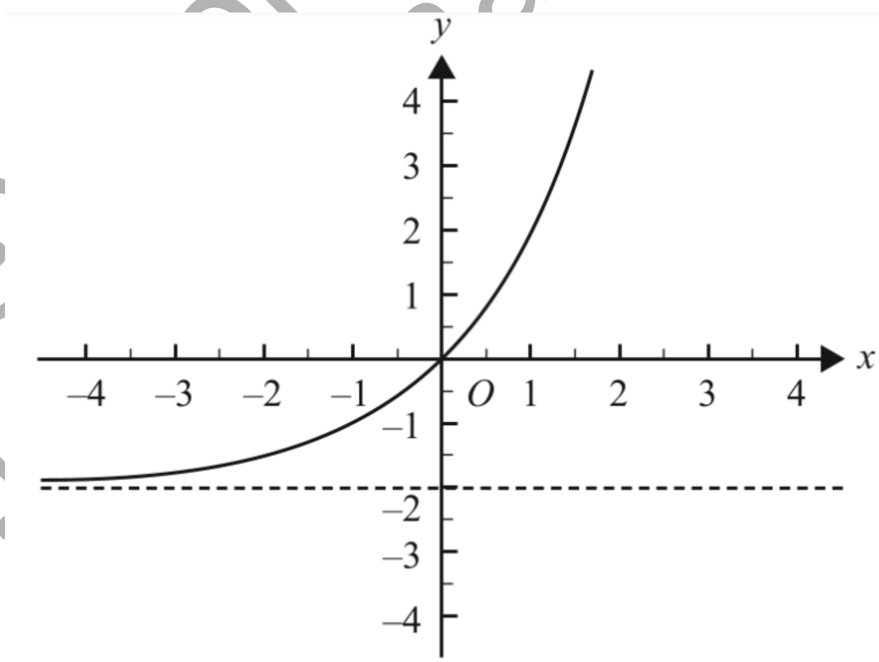


g. Find α in degrees, correct to two decimal places.

h. Hence or otherwise, find the length of time, to the nearest minute, during which the boat at B is visible.

2017 - Exam 2 - Section 2 - Question 4 - Part g-i

Let $f : R \to R, f(x) = 2^{x+1} - 2$. Part of the graph of f is shown below



The functions of g_k, where $k \in R^+$, are defined with domain R such that $g_k(x) = 2e^{kx} - 2$.

e. Find the value of k such that $g_k(x) = f(x)$.

f. Find the rule for the inverse functions g_k^{-1} of g_k, where $k \in R^+$.

g. **i.** Describe the transformation that maps the graph of g_1 onto the graph of g_k.

ii. Describe the transformation that maps the graph of g_1^{-1} onto the graph of g_k^{-1}.

h. The lines L_1 and L_2 are the tangents at the origin to the graphs of g_k and g_k^{-1} respectively.

Find the value(s) of k for which the angle between L_1 and L_2 is $30°$.

i. Let p be the value of k for which $g_k(x) = g_k{}^{-1}(x)$ has only one solution.

i. Find p.

ii. Let $A(k)$ be the area bounded by the graphs of g_k and g_k^{-1} for all $k > p$. State the smallest value of b such that $A(k) < b$.

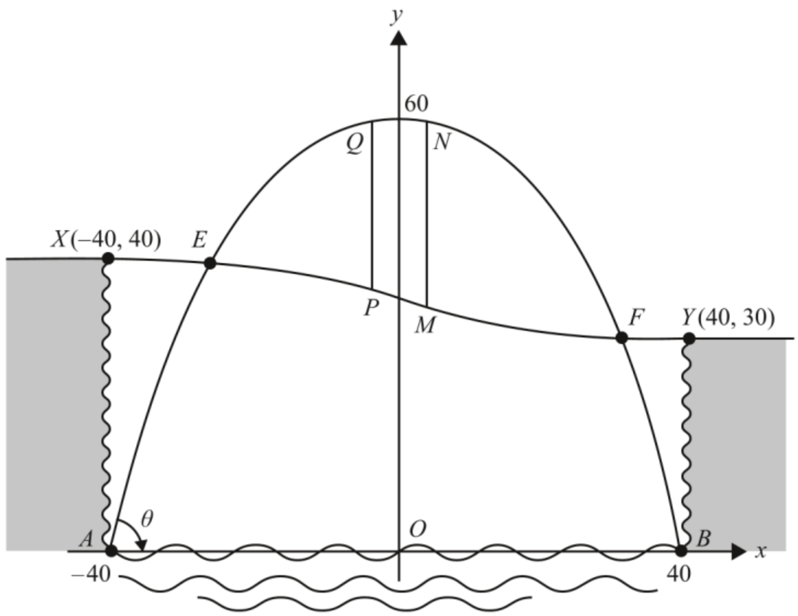
2015 - Exam 2 - Section 1 - Question 2 - Part a-f

A city is located on a river that runs through a gorge.

The gorge is $80\ \mathrm{m}$ across, $40\ \mathrm{m}$ high on one side and $30\ \mathrm{m}$ high on the other side.

A bridge is to be built that crosses the river and the gorge.

A diagram for the design of the bridge is shown below.



The main frame of the bridge has the shape of a parabola. The parabolic frame is modelled by $y = 60 - \frac{3}{80}x^2$ and is connected to concrete pads at $B(40, 0)$ and $A(-40, 0)$.

The road across the gorge is modelled by a cubic polynomial function.

a. Find the angle, θ, between the tangent to the parabolic frame and the horizontal at the point $A(-40, 0)$ to the nearest degree.

The road from X to Y across the gorge has gradient zero at $X(-40, 40)$ and at $Y(40, 30)$, and has equation $y = \dfrac{x^3}{25600} - \dfrac{3x}{16} + 35$.

b. Find the maximum downwards slope of the road. Give your answer in the form $-\dfrac{m}{n}$ where m and n are positive integers.

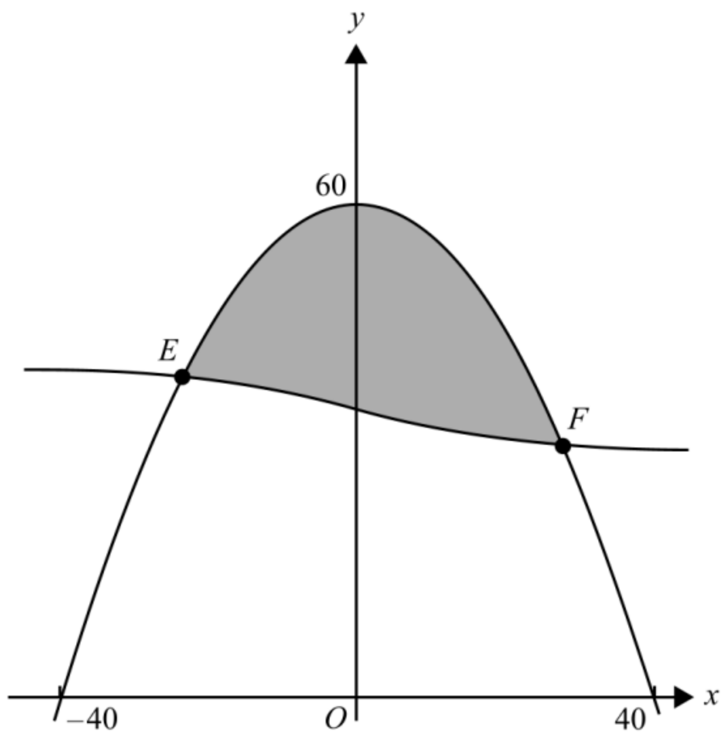
Two vertical supporting columns, MN and PQ, connect the road with the parabolic frame. The supporting column, MN, is at the point where the vertical distance between the road and the parabolic frame is a maximum.

c. Find the coordinates (u, v) of the point M, stating your answers correct to two decimal places.

The second supporting column, PQ, has its lowest point at $P(-u, w)$.

d. Find, correct to two decimal places, the value of w and the lengths of the supporting columns MN and PQ.

For the opening of the bridge, a banner is erected on the bridge, as shown by the shaded region in the diagram below.

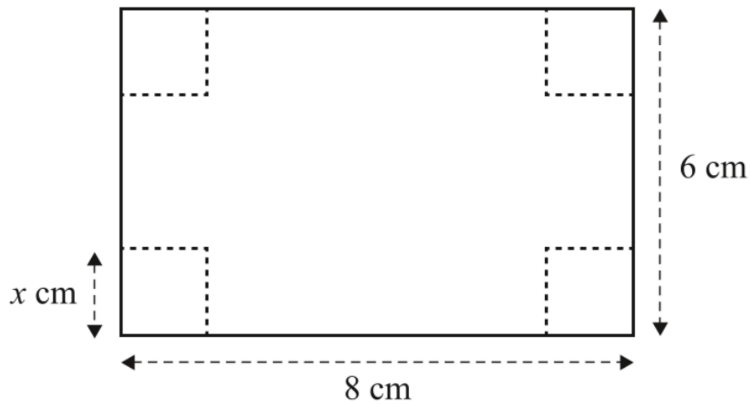


e. Find the x-coordinates, correct to two decimal places, of E and F, the points at which the road meets the parabolic frame of the bridge.

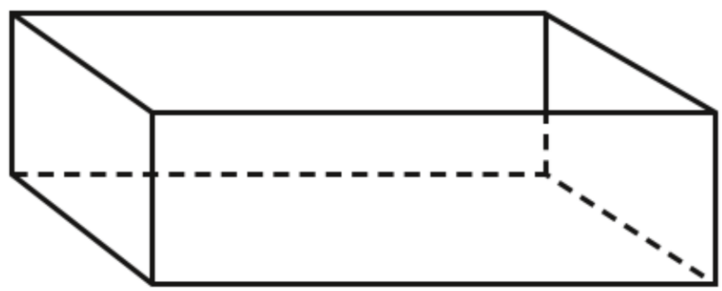
f. Find the area of the banner (shaded region), giving your answer to the nearest square metre.

2014 - Exam 2 - Section 1 - Question 15

Zoe has a rectangular piece of cardboard that is 8 cm long and 6 cm wide. Zoe cuts squares of side length x centimetres from each of the corners of the cardboard, as shown in the diagram below.



Zoe turns up the sides to form an open box.



The value of x for which the volume of the box is a maximum to closest to

A. 0.8

B. 1.1

E. 1.6

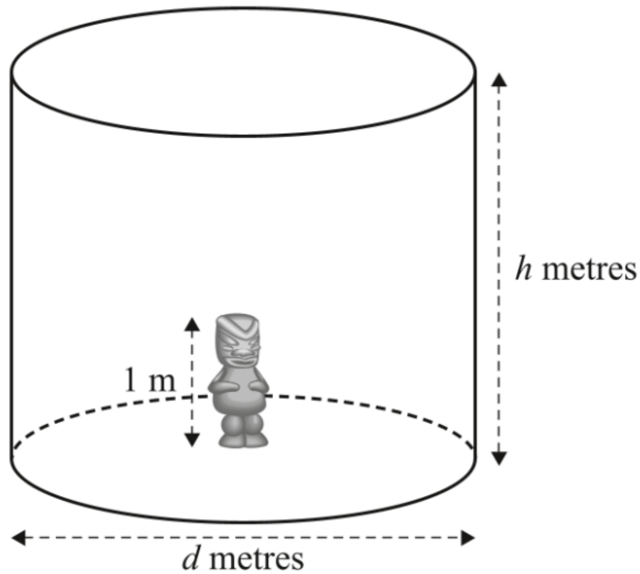
D. 2.0

E. 3.6

2014 - Exam 2 - Section 2 - Question 2 - Part a-h

On 1 January 2010, Tasmania Jones was walking through an ice-covered region of Greenland when he found a large ice cylinder that was made a thousand years ago by the Vikings.

A statue was inside the ice cylinder. The statue was $1\ \mathrm{m}$ tall and its base was at the centre of the base of the cylinder.



The cylinder had a height of h metres and a diameter of d metres. Tasmania Jones found that the volume of the cylinder was $216\ \mathrm{m}^3$. At that time, 1 January 2010, the cylinder had not changed in a thousand years. It was exactly as it was when the Vikings made it.

a. Write an expression for h in terms of d.

b. Show that the surface area of the cylinder excluding the base, S square metres, is given by the rule $S = \dfrac{\pi d^2}{4} + \dfrac{864}{d}$.

Tasmania found that the Vikings made the cylinder so that S is a minimum.

c. Find the value of d for which S is a minimum and find this minimum value of S.

d. Find the value of h when S is a minimum.

On 1 January 2010, Tasmania believed that due to recent temperature changes in Greenland, the ice of the cylinder had just started melting. Therefore, he decided to return on 1 January each year to measure the ice cylinder. He observes that the volume of the ice cylinder decreases by a constant rate of $10\ \mathrm{m}^3$ per year. Assume that the cylindrical shape is retained and $d = 2h$ at the beginning and as the cylinder melts.

e. Write down an expression for V in terms of h.

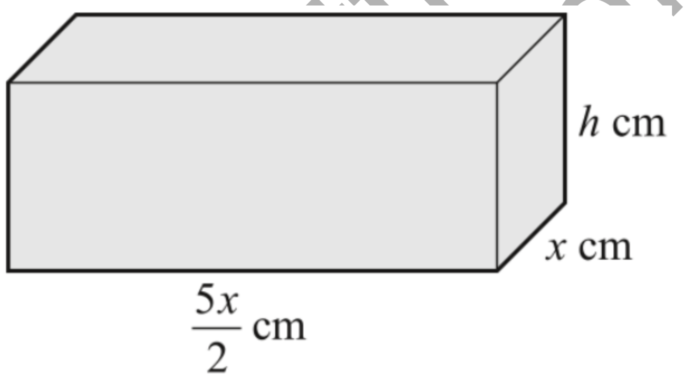
f. Find $\dfrac{dh}{dt}$ in terms of h.

g. Find the rate at which the height of the cylinder will be decreasing when the top of the statue is just exposed.

h. Find the year in which the top of the statue will just be exposed. (Assuming that the melting started on 1 January 2010)

2012 - Exam 2 - Section 2 - Question 1 - Part a-d

A solid block in the shape of a rectangular prism has a base of width x cm. The length of the base is two-and-a-half times the width of the base.



The block has a total surface area of 6480sqcm.

a. Show that if the height of the block is h cm, $h = \dfrac{6480 - 5x^2}{7x}$.

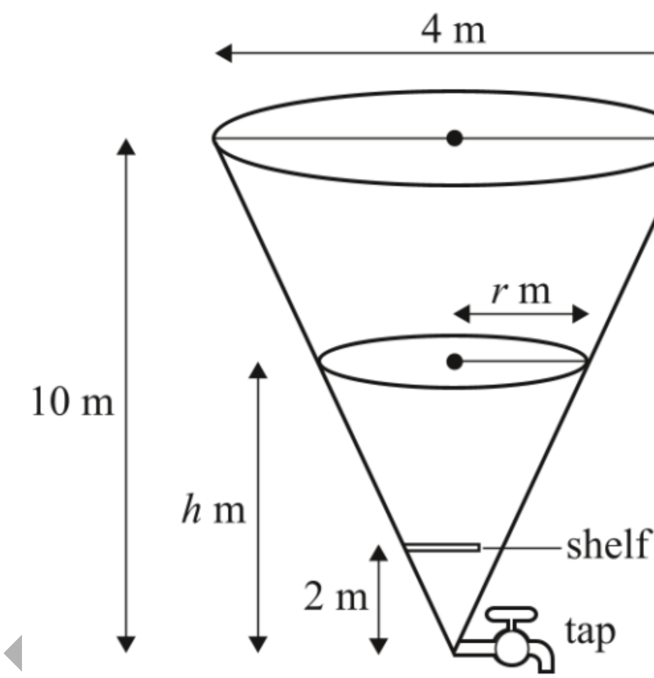
b. The volume, V cm^3, of the block is given by $V(x) = \dfrac{5x\left(6480 - 5x^2\right)}{14}$. Given that $V(x) > 0$ and $x > 0$, find the possible values of x.

c. Find $\dfrac{dV}{dx}$, expressing your answer in the form $\dfrac{dV}{dx} = ax^2 + b$, where a and b are real numbers.

d. Find the exact values of x and h if the block is to have maximum value.

2012 - Exam 2 - Section 2 - Question 4 - Part a-f

Tasmania Jones is in the jungle, searching for the Quetzalotl tribe's valuable emerald that has been stolen and hidden by a neighbouring tribe. Tasmania has heard that the emerald has been hidden in a tank shaped like an inverted cone, with a height of 10 metres and a diameter of 4 metres (as shown below).



The emerald is on a shelf. The tank has a poisonous liquid in it.

a. If the depth of the liquid in the tank is h metres

i. find the radius, r metres, of the surface of the liquid in terms of h

ii. show that the volume of the liquid in the tank is $\frac{\pi h^3}{75}$ m^3.

The tank has a tap at its base that allows the liquid to run out of it. The tank is initially full. When the tap is turned on, the liquid flows out of the tank at such a rate that the depth, h metres, of the liquid in the tank is given by

$$h = 10 + \frac{1}{1600}\left(t^3 - 1200t\right),$$

where t minutes is the length of time after the tap is turned on until the tank is empty.

b. Show that the tank is empty when $t = 20$.

c. When $t = 5$ minutes, find

i. the depth of the liquid in the tank

ii. the rate at which the volume of the liquid is decreasing, correct to one decimal place.

d. The shelf on which the emerald is placed is 2 metres above the vertex of the cone.

From the moment the liquid starts to flow from the tank, find how long, in minutes, it takes until $h = 2$. (Give your answer correct to one decimal place.)

e. As soon as the tank is empty, the tap turns itself off and poisonous liquid starts to flow into the tank at a rate of $0.2\ \mathrm{m}^3/$ minute.

How long, in minutes, after the tank is first empty will the liquid once again reach a depth of 2 metres?

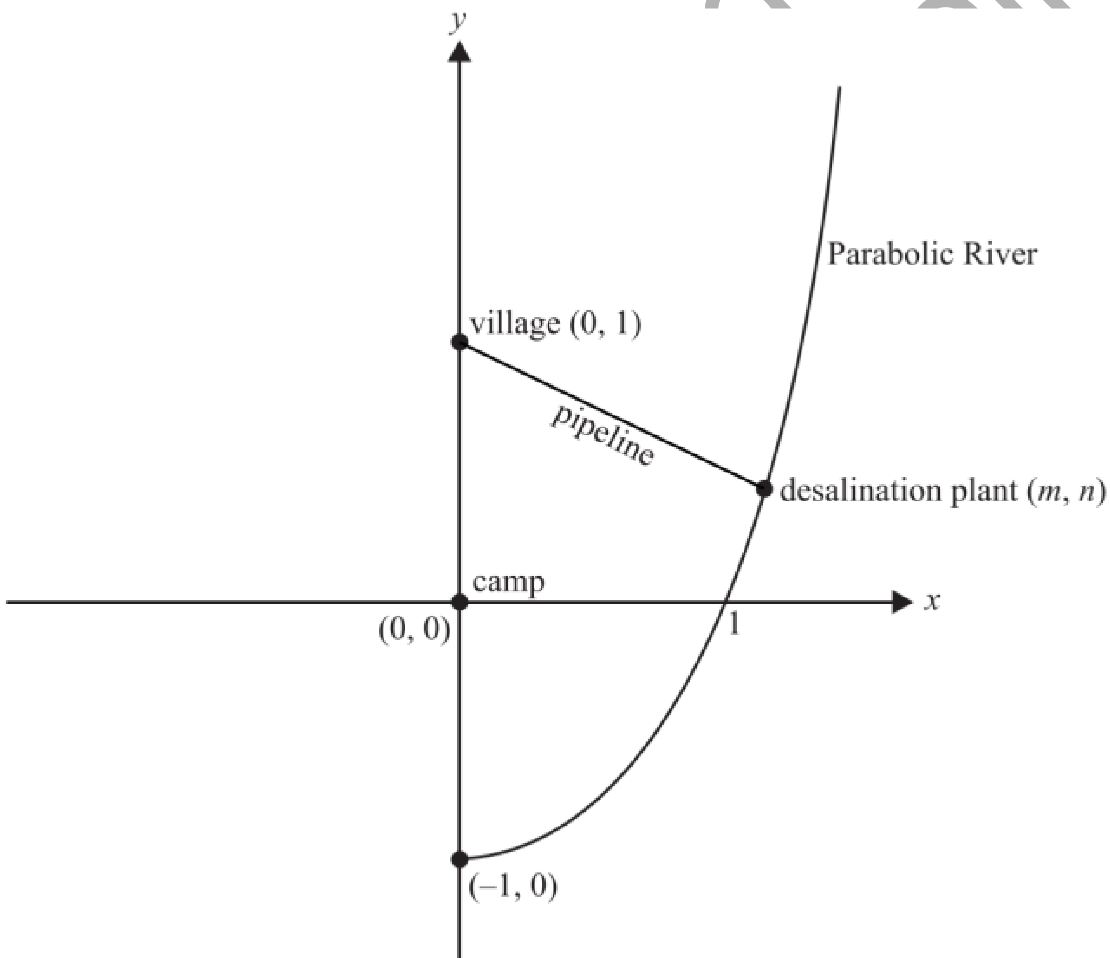
f. In order to obtain the emerald, Tasmania Jones enters the tank using a vine to climb down the wall of the tank as soon as the depth of the liquid is first 2 metres. He must leave the tank before the depth is again greater than 2 metres.

Find the length of time, in minutes, correct to one decimal place, that Tasmania Jones has from the time he enters the tank to the time he leaves the tank.

2011 - Exam 2 - Section 2 - Question 4 - Part a-f

Deep in the South American jungle, Tasmania Jones has been working to help the Quetzacotl tribe to get drinking water from the very salty water of the Parabolic River. The river follows the curve with equation $y = x^2 - 1, x \geq 0$ as shown below. All lengths are measured in kilometres.

Tasmania has his camp site at $(0, 0)$ and the Quetzacotl tribe's village is at $(0, 1)$. Tasmania builds a desalination plant, which is connected to the village by a straight pipeline.



a. If the desalination plant is at the point (m, n) show that the length, L kilometres, of the straight pipeline that carries the water from the desalination plant to the village is given by

$$L = \sqrt{m^4 - 3m^2 + 4}.$$

b. If the desalination plant is built at the point on the river that is closest to the village

i. find $\frac{dL}{dm}$ and hence find the coordinates of the desalination plant

ii. find the length, in kilometres, of the pipeline from the desalination plant to the village.

The desalination plant is actually built at $\left(\frac{\sqrt{7}}{2}, \frac{3}{4}\right)$.

If the desalination plant stops working, Tasmania needs to get to the plant in the minimum time.

Tasmania runs in a straight line from his camp to a point (x, y) on the river bank where $x \leq \frac{\sqrt{7}}{2}$. He then swims up the river to the desalination plant.

Tasmania runs from his camp to the river at $2\ \mathrm{km}$ per hour. The time that he takes to swim to the desalination plant is proportional to the difference between the y-coordinates of the desalination plant and the point where he enters the river.

c. Show that the total time taken to get to the desalination plant is given by

$$T = \frac{1}{2}\sqrt{x^4 - x^2 + 1} + \frac{1}{4}k\left(7 - 4x^2\right)$$

hours where k is a positive constant of proportionality.

The value of k varies from day to day depending on the weather conditions.

d. If $k = \dfrac{1}{2\sqrt{13}}$

i. find $\dfrac{dT}{dx}$

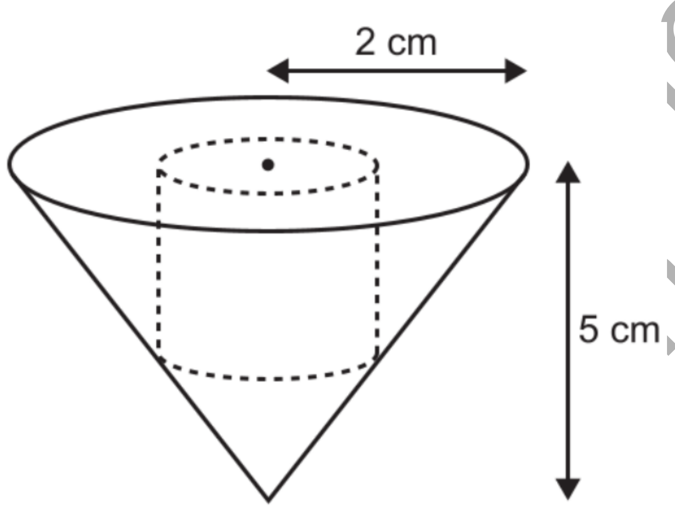
ii. hence find the coordinates of the point where Tasmania should reach the river if he is to get to the desalination plant in the minimum time.

e. On one particular day, the value of k is such that Tasmania should run directly from his camp to the point $(1, 0)$ on the river to get to the desalination plant in the minimum time. Find the value of k on that particular day.

f. Find the values of k for which Tasmania should run directly from his camp towards the desalination plant to reach it in the minimum time.

2010 - Exam 1 - Section 1 - Question 11 - Part a-c

A cylinder fits exactly in a right circular cone so that the base of the cone and one end of the cylinder are in the same plane as shown in the diagram below. The height of the cone is 5 cm and the radius of the cone is 2 cm. The radius of the cylinder is $r \text{ cm}$ and the height of the cylinder is $h \text{ cm}$.



For the cylinder inscribed in the cone as shown above

a. find h in terms of r

The total surface area, $S \text{ cm}^2$, of a cylinder of height $h \text{ cm}$ and radius $r \text{ cm}$ is given by the formula

$$S = 2\pi rh + 2\pi r^2.$$

b. find S in terms of r

c. find the value of r for which S is a maximum.